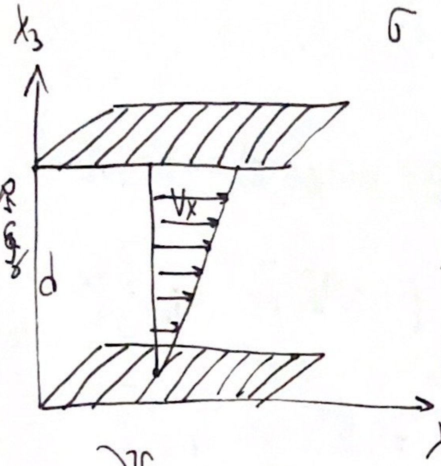
Коэф-ты первой и второй вязкости

$$\Pi = -p\underline{I} + \hat{\sigma} = -p\underline{I} + \underbrace{a\dot{S} + b\underline{I}}_{\hat{\sigma}} \quad (\hat{\sigma} = 2\mu\dot{S} + \bar{\mu} S_p \dot{S}\underline{I})$$



d-расст. между пластинами.

Между пласт-ми (беск-ми) нах-ся жидкость.

Нижняя пласт-на ($x_3 = 0$) нах-ся в покое

Верхняя пласт-на движется со ск-тью v в напр. x_1

Теч-е плоское $v_2 = v_3 = 0$. $v_1(x_3) = v\frac{x_3}{d}$ (из опыта); касат. напр-е:

$\tau = \frac{F}{S} = \mu\frac{v}{d}$ (напряж. соот. силы для передвиж. пласт-ны)

$\tau_{13} = \mu\frac{\partial v_1}{\partial x_3}$; v_1 линейно зависит от x_3 только $\Rightarrow \operatorname{div} v = 0$

$$\hat{\sigma}_{13} = a\cdot\frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \underbrace{\frac{\partial v_3}{\partial x_1}}_{0}\right) = \mu\frac{\partial v_1}{\partial x_3} \Rightarrow \frac{a}{2}\frac{\partial v_1}{\partial x_3} = \mu\frac{\partial v_1}{\partial x_3} \Rightarrow a = 2\mu$$

μ – коэф-т первой вязкости. ζ – коэф-т второй вязкости

Ур-е энергии

$$\frac{d}{dt}\int_V \rho\left(\varepsilon + \frac{v^2}{2}\right)dV = \oint\!\!\oint_\Sigma (pn, v)\,d\Sigma + \oint\!\!\oint_\Sigma (\hat{\sigma}n, v)\,d\Sigma; \quad \frac{d}{dt}(\rho dV) = 0$$ – ур-е неразр. в Лагр. к.

$$\int_V \rho\frac{d}{dt}\left(\varepsilon + \frac{v^2}{2}\right)dV = -\int_V \operatorname{div}(pv)\,dV + \int_V \operatorname{div}(\hat{\sigma}v)\,dV$$

$\rho\frac{d}{dt}\left(\varepsilon + \frac{v^2}{2}\right) = -\operatorname{div}(pv) + \operatorname{div}(\hat{\sigma}v)$ – ур-е энергии

умнож. скалярно на v

$\rho\frac{dv}{dt} = -\operatorname{grad} p + \operatorname{div}\hat{\sigma}$: (ур-е движ. в Лагр. к.)

$(\operatorname{div}\Pi = \operatorname{div}(-p\underline{I} + \hat{\sigma})$; $\Pi = -p\underline{I} + \hat{\sigma}$

$\Rightarrow \rho\frac{d}{dt}\left(\frac{v^2}{2}\right) = (-\operatorname{grad} p, v) + (\operatorname{div}\hat{\sigma}, v)$

$\rho\frac{d}{dt}\varepsilon + (-\operatorname{grad} p, v) + (\operatorname{div}\hat{\sigma}, v) = -\operatorname{div}(pv) + \operatorname{div}(\hat{\sigma}v)$;

$$\operatorname{div}(\hat{\sigma}v) = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\sum_{k=1}^{3}\hat{\sigma}_{ik}v_k = \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial}{\partial x_i}(\hat{\sigma}_{ik}v_k) =$$

$$= \sum_{i=1}^{3}\sum_{k=1}^{3}\left(v_k\frac{\partial}{\partial x_i}\hat{\sigma}_{ik} + \hat{\sigma}_{ik}\frac{\partial}{\partial x_i}v_k\right) =$$

$$= \sum_{k=1}^{3} v_k \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\hat{\sigma}_{ik} + \sum_{i,k}\hat{\sigma}_{ik}\frac{\partial v_k}{\partial x_i} =$$

$$= \sum_{k=1}^{3} v_k (\mathrm{div}\,\hat\sigma)_k + \sum_{i,k} \hat\sigma_{ik}\frac{\partial v_k}{\partial x_i} = (v, \mathrm{div}\,\hat\sigma) + \sum_{i,k} \hat\sigma_{ik}\frac{\partial v_k}{\partial x_i};$$

$$\rho\frac{d}{dt}\varepsilon + \rho\frac{d}{dt}\left(\frac{v^2}{2}\right) = -p\,\mathrm{div}\,v - (v, \mathrm{grad}\,p) + (v, \mathrm{div}\,\hat\sigma) + \sum_{i,k}\hat\sigma_{ik}\frac{\partial v_k}{\partial x_i}$$

Ур-е движ-я: $\rho\frac{dv}{dt} = \mathrm{div}\,\Pi;\ \rho\frac{dv}{dt} = -\mathrm{grad}\,p + \mathrm{div}\,\hat\sigma \Rightarrow$

$$\Rightarrow \rho\frac{d}{dt}\left(\frac{v^2}{2}\right) = -(\mathrm{grad}\,p, v) + (\mathrm{div}\,\hat\sigma, v);\ \rho\frac{d}{dt}\varepsilon \cancel{-(\mathrm{grad}\,p, v)} + \cancel{(\mathrm{div}\,\hat\sigma, v)} = -p\,\mathrm{div}\,v -$$

$$-\cancel{(v, \mathrm{grad}\,p)} + \cancel{(v, \mathrm{div}\,\hat\sigma)} + \sum_{i,k}\hat\sigma_{ik}\frac{\partial v_k}{\partial x_i};\quad \rho\frac{d}{dt}\varepsilon = -p\,\mathrm{div}\,v + \sum_{i,k}\hat\sigma_{ik}\frac{\partial v_k}{\partial x_i}$$

$$\hat\sigma_{ik} = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \zeta\delta_{ik}\,\mathrm{div}\,v - \frac{2}{3}\mu\,\delta_{ik}\,\mathrm{div}\,v;\ \hat\sigma = 2\mu\dot S + \bar\mu\, S_p \dot S\, I$$

$$\sum_{i,k}\hat\sigma_{ik}\frac{\partial v_k}{\partial x_i} = \sum_{i,k}\left(\mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \zeta\delta_{ik}\,\mathrm{div}\,v - \frac{2}{3}\mu\,\delta_{ik}\,\mathrm{div}\,v\right)\frac{\partial v_k}{\partial x_i};\ \sum_{i,k}\delta_{ik}\frac{\partial v_k}{\partial x_i} = \mathrm{div}\,v;$$

$$\sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\frac{\partial v_k}{\partial x_i} = \sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\frac{\partial v_i}{\partial x_k} = S;\ 2S = \sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right);$$

$$S = \frac{1}{2}\sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right);\ \sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\frac{\partial v_k}{\partial x_i} = \sum_{i,k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right)$$

$$\frac{\partial v_k}{\partial x_i} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right);\ = \sum_{i,k}\mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\,\mathrm{div}\,v\right)\frac{\partial v_k}{\partial x_i} + \zeta(\mathrm{div}\,v)^2 =$$

$$= \sum_{i,k}\mu\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\,\mathrm{div}\,v\right)\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \pm \frac{2}{3}\delta_{ik}\,\mathrm{div}\,v\right) + \zeta(\mathrm{div}\,v)^2 =$$

$$= \sum_{i,k}\frac{\mu^2}{2\mu}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\,\mathrm{div}\,v\right)^2 + \underbrace{\sum_{i,k}\frac{\mu}{2}\left(\frac{\partial v_i}{\partial x_k} + \underset{=0}{\frac{\partial v_k}{\partial x_i}} - \frac{2}{3}\delta_{ik}\,\mathrm{div}\,v\right)\frac{2}{3}\delta_{ik}\,\mathrm{div}\,v}_{(S_p\tau = 0)} + \zeta(\mathrm{div}\,v)^2$$

$\rho\frac{d}{dt}\varepsilon = -p\,\mathrm{div}\,v + \zeta(\mathrm{div}\,v)^2 + \frac{1}{2\mu}\tau^2$ – ур-е для внутр энергии, где μ – коэф. I вяз-ти, $\mu \geq 0$
ζ – коэф II вяз-ти, $\zeta \geq 0$. Внутр. энергия увелич. за счёт слагаемых $\underset{\text{объёмная деф-я}}{\zeta(\mathrm{div}\,v)^2}$, $\frac{1}{2\mu}\tau^2$ – сдвиговая деф-я